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A note on the non-diagonal K-matrices for the trigonometric $A^{(1)}_{n-1}$ vertex model

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Abstract

This note presents explicit matrix expressions of a class of recently-discovered non-diagonal K-matrices for the trigonometric $A^{(1)}_{n-1}$ vertex model. From these explicit expressions, it is easily seen that in addition to a *discrete* (positive integer) parameter l, $1 \leq l \leq n$, the K-matrices contain $n+1$ (or n) continuous free boundary parameters.

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Inspired from the observation [1] that the generic non-diagonal solutions (or K-matrices) [2, 3] of the reflection equation for the spin-$\frac{1}{2}$ XXZ model are decomposed into the product of intertwiner-matrices and diagonal face-type K-matrix, in [4] an intertwiner-matrix approach was developed and used to construct a class of non-diagonal solutions of the reflection equation for the trigonometric $A^{(1)}_{n-1}$ vertex model. There the K-matrices were expressed in terms of the intertwiner-matrix and a diagonal matrix. To fully realize the application of the solutions obtained in [4], it may be useful to write them in *explicit* and *familiar* matrix form. The purpose of this note is to provide such explicit expressions. From these expressions it is easily seen that in addition to a *discrete* (positive integer) parameter l, $1 \leq l \leq n$, the

solutions we constructed in [4] contain $n+1$ (or n) continuous free boundary parameters and have $3n-2$ (or $2n-1$) non-vanishing matrix elements.

Our starting point in [4] is the trigonometric R-matrix associated with the n-dimensional representation of $A^{(1)}_{n-1}$ given in [5, 6]:

$$R(u)=\sum_{\alpha=1}^{n}R^{\alpha\alpha}_{\alpha\alpha}(u)E_{\alpha\alpha}\otimes E_{\alpha\alpha}+\sum_{\alpha\neq\beta}\left\{R^{\alpha\beta}_{\alpha\beta}(u)E_{\alpha\alpha}\otimes E_{\beta\beta}+R^{\beta\alpha}_{\alpha\beta}(u)E_{\beta\alpha}\otimes E_{\alpha\beta}\right\}, \qquad (1)$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k=\delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{\alpha\beta}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)\,e^{-i\eta}}{\sin(u+\eta)}, & \alpha>\beta,\\ 1, & \alpha=\beta,\\ \frac{\sin(u)\,e^{i\eta}}{\sin(u+\eta)}, & \alpha<\beta, \end{cases} \qquad (2)$$

$$R^{\beta\alpha}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(\eta)\,e^{iu}}{\sin(u+\eta)}, & \alpha>\beta,\\ 1, & \alpha=\beta,\\ \frac{\sin(\eta)\,e^{-iu}}{\sin(u+\eta)}, & \alpha<\beta. \end{cases} \qquad (3)$$

Here η is the so-called crossing parameter. In addition to the quantum Yang-Baxter equation, the R-matrix satisfies the following unitarity, crossing-unitarity and quasi-classical relations:

$$\text{Unitarity}: \qquad R_{12}(u)R_{21}(-u)=\mathrm{id}, \qquad (4)$$

$$\text{Crossing-unitarity}: \qquad R^{t_2}_{12}(u)M_2^{-1}R^{t_2}_{21}(-u-n\eta)M_2=\frac{\sin(u)\sin(u+n\eta)}{\sin(u+\eta)\sin(u+n\eta-\eta)}\,\mathrm{id}, \qquad (5)$$

$$\text{Quasi-classical property}: \qquad R_{12}(u)|_{\eta\to 0}=\mathrm{id}. \qquad (6)$$

Here $R_{21}(u)=P_{12}R_{12}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes the transposition in the i-th space. The crossing matrix M is a diagonal $n\times n$ matrix with elements

$$M_{\alpha\beta}=M_\alpha\delta_{\alpha\beta},\quad M_\alpha=e^{-2i\alpha\eta},\ \alpha=1,\cdots,n. \qquad (7)$$

Boundary K-matrices $K^-(u)$ and $K^+(u)$, which give rise to integrable boundary conditions of an open chain on the right and left boundaries, respectively, satisfy the reflection and dual reflection equations [7, 8]:

$$\begin{aligned} &R_{12}(u_1-u_2)K^-_1(u_1)R_{21}(u_1+u_2)K^-_2(u_2)\\ &\qquad =K^-_2(u_2)R_{12}(u_1+u_2)K^-_1(u_1)R_{21}(u_1-u_2), \end{aligned} \qquad (8)$$

$$\begin{aligned} &R_{12}(u_2-u_1)K^+_1(u_1)\,M_1^{-1}\,R_{21}(-u_1-u_2-n\eta)\,M_1\,K^+_2(u_2)\\ &\qquad =M_1\,K^+_2(u_2)R_{12}(-u_1-u_2-n\eta)\,M_1^{-1}\,K^+_1(u_1)R_{21}(u_2-u_1). \end{aligned} \qquad (9)$$

Different integrable boundary conditions are described by different solutions $K^-(u)$ ($K^+(u)$) to the (dual) reflection equation [7, 3].

Let us briefly recall some of the results in [4]. Let $\{\epsilon_i \mid i = 1, 2, \cdots, n\}$ be the orthonormal basis of the vector space $\mathbb{C}^n$ such that $\langle \epsilon_i,\ \epsilon_j \rangle = \delta_{ij}$. For a generic vector $\lambda \in \mathbb{C}^n$, define

$$\lambda_i = \langle \lambda, \epsilon_i \rangle, \quad |\lambda| = \sum_{k=1}^{n} \lambda_k, \quad i = 1, \cdots, n. \tag{10}$$

Let us introduce an $n \times n$ matrix $\Phi(u;\lambda)$ which depends on the spectrum parameter u and λ. The non-vanishing matrix elements of $\Phi(u;\lambda)$ are given by

$$\begin{pmatrix} e^{i\eta f_1(\lambda)} & & & & & & & e^{i\eta F_n(\lambda)+\rho}e^{2iu} \\ e^{i\eta F_1(\lambda)} & e^{i\eta f_2(\lambda)} & & & & & & \\ & e^{i\eta F_2(\lambda)} & \ddots & & & & & \\ & & \ddots & e^{i\eta f_j(\lambda)} & & & & \\ & & & e^{i\eta F_j(\lambda)} & \ddots & & & \\ & & & & \ddots & e^{i\eta f_{n-1}(\lambda)} & & \\ & & & & & e^{i\eta F_{n-1}(\lambda)} & & e^{i\eta f_n(\lambda)} \end{pmatrix}. \tag{11}$$

Here ρ is a complex constant with regard to u and λ, and $\{f_i(\lambda)|i = 1, \ldots, n\}$ and $\{F_i(\lambda)|i = 1, \ldots, n\}$ are linear functions of λ:

$$f_i(\lambda) = \sum_{k=1}^{i-1} \lambda_k - \lambda_i - \frac{1}{2}|\lambda|, \quad i = 1, \ldots, n, \tag{12}$$

$$F_i(\lambda) = \sum_{k=1}^{i} \lambda_k - \frac{1}{2}|\lambda|, \quad i = 1, \ldots, n-1, \tag{13}$$

$$F_n(\lambda) = -\frac{3}{2}|\lambda|. \tag{14}$$

The determinant of $\Phi(u;\lambda)$ is [4]

$$\mathrm{Det}\,(\Phi(u;\lambda)) = e^{i\eta \sum_{k=1}^{n} \frac{n-2(k+1)}{2}\lambda_k}\,(1 - (-1)^n e^{2iu+\rho}). \tag{15}$$

For a generic $\rho \in \mathbb{C}$ this determinant is not vanishing and thus the inverse of $\Phi(u;\lambda)$ exists. Associated to a positive integer l ($1 \le l \le n$), let us introduce a diagonal matrix

$$D^{(l)}(u) = \mathrm{Diag}(k_1^{(l)}(u), \ldots, k_n^{(l)}(u)), \tag{16}$$

where $\{k_i^{(l)}(u)|i = 1, \ldots, n\}$ are

$$k_j^{(l)}(u) = \begin{cases} 1, & 1 \le j \le l, \\ \frac{\sin(\xi-u)}{\sin(\xi+u)}e^{-2iu}, & l+1 \le j \le n. \end{cases} \tag{17}$$

Here ξ is free complex parameter. Then one can define the non-diagonal K-matrices $\{K^{(l)}(u)|l = 1,\ldots,n\}$ associated with $\{D^{(l)}(u)|l = 1,\ldots,n\}$ and $\Phi(u;\lambda)$ as follows [4]:

$$K^{(l)}(u) = \Phi(u;\lambda)D^{(l)}(u)\left\{\Phi(-u;\lambda)\right\}^{-1}, \quad l = 1,\ldots,n. \tag{18}$$

It has been shown in [4] that the matrix $\Phi(u;\lambda)$ given by (11) is the intertwiner-matrix which intertwines two trigonometric R-matrices, and thus the non-diagonal K-matrices $\{K^{(l)}(u)\}$ given by (18) solve the reflection equation (8) for the trigonometric $A^{(1)}_{n-1}$ vertex model. Moreover, (18) implies that the K-matrices satisfy the regular condition $K^{(l)}(0) = \mathrm{id}$, $l = 1,\ldots,n$, and boundary unitarity relation $K^{(l)}(u)K^{(l)}(-u) = \mathrm{id}$, $l = 1,\ldots,n$.

Through a tedious calculation for $n = 2,3,4,5$ with the help of Mathematica program, we reconfirm the following properties for the non-diagonal K-matrices (18): $K^{(l)}(u)$ $(l = 1,\ldots,n-1)$ depend on $n+1$ continuous free parameters ξ, $\{\lambda_i|i = 1,\ldots,n-1\}$ and ρ, and have $3n-2$ non-vanishing matrix elements (c.f. [9, 10]); $K^{(n)}(u)$ depends on n continuous free parameters $\{\lambda_i|i = 1,\ldots,n-1\}$ and ρ, and has $2n-1$ non-vanishing matrix elements (c.f. [9, 10]). The dependence on λ_n *disappears* in the final expressions of the K-matrices although it appears in the expression of $\Phi(u;\lambda)$. The above properties are expected to hold for generic n. In the rational limit, the trigonometric K-matrices (18) reduce to those corresponding to the rational $A^{(1)}_{n-1}$ vertex model [11, 12] with a special choice of the spectral-independent similarity transformation matrix.

In the following, we give the explicit matrix expressions of the K-matrices (18) for the cases $n = 3,4$.

The $A^{(1)}_2$ case:

There are three types of K-matrices for the trigonometric $A^{(1)}_2$ model.

- For the K-matrix $K^{(1)}(u)$, the 7 non-vanishing matrix elements $K(u)^k_j$ are given by:

$$\begin{aligned}
K(u)^1_1 &= \frac{e^{2iu}}{e^{2iu}+e^{\rho}}\left(1 - e^{\rho}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right), \; K(u)^1_2 = \frac{e^{-2i\eta\lambda_1+\rho}}{e^{2iu}+e^{\rho}}\left(1 + e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)^1_3 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}}{e^{2iu}+e^{\rho}}\left(1 + e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)^2_1 &= \frac{e^{2i\eta\lambda_1+i(u+\xi)}\sin 2u}{(e^{2iu}+e^{\rho})\sin(u+\xi)}, \; K(u)^2_2 = \frac{1}{e^{2iu}+e^{\rho}}\left(e^{\rho} - \frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)^2_3 &= -\frac{e^{-2i\eta\lambda_2-i(u-\xi)+\rho}\sin 2u}{(e^{2iu}+e^{\rho})\sin(u+\xi)}, \; K(u)^3_3 = e^{-2iu}\frac{\sin(\xi-u)}{\sin(\xi+u)}.
\end{aligned} \tag{19}$$

- For the K-matrix $K^{(2)}(u)$, the 7 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}}{e^{2iu}+e^{\rho}}\left(1-e^{\rho}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^1 = \frac{e^{-2i\eta\lambda_1+\rho}}{e^{2iu}+e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_3^1 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}}{e^{2iu}+e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_2^2 &= 1,\\
K(u)_1^3 &= -\frac{e^{2i\eta(\lambda_1+\lambda_2)+i(u+\xi)}\sin 2u}{(e^{2iu}+e^{\rho})\sin(u+\xi)},\ K(u)_2^3 = \frac{e^{2i\eta\lambda_2+i(u+\xi)}\sin 2u}{(e^{2iu}+e^{\rho})\sin(u+\xi)},\\
K(u)_3^3 &= \frac{1}{e^{2iu}+e^{\rho}}\left(e^{\rho}-\frac{\sin(u-\xi)}{\sin(u+\xi)}\right).
\end{aligned}
\tag{20}
$$

- For the K-matrix $K^{(3)}(u)$, the 5 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}+e^{4iu+\rho}}{e^{2iu}+e^{\rho}},\ K(u)_2^1 = -\frac{e^{-2i\eta\lambda_1+\rho}\left(e^{4iu}-1\right)}{e^{2iu}+e^{\rho}},\\
K(u)_3^1 &= \frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}\left(e^{4iu}-1\right)}{e^{2iu}+e^{\rho}},\ K(u)_2^2 = K(u)_3^3 = 1.
\end{aligned}
\tag{21}
$$

The $A_3^{(1)}$ case:

There are four types of K-matrices for the trigonometric $A_3^{(1)}$ model.

- For the K-matrix $K^{(1)}(u)$, the 10 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}}{e^{2iu}-e^{\rho}}\left(1+e^{\rho}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^1 = -\frac{e^{-2i\eta\lambda_1+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_3^1 &= \frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_4^1 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2+\lambda_3)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_1^2 = \frac{e^{2i\eta\lambda_1+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\\
K(u)_2^2 &= -\frac{1}{e^{2iu}-e^{\rho}}\left(e^{\rho}+\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_3^2 = \frac{e^{-2i\eta\lambda_2-i(u-\xi)+\rho}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\\
K(u)_4^2 &= -\frac{e^{-2i\eta(\lambda_2+\lambda_3)-i(u-\xi)+\rho}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\ K(u)_3^3 = K(u)_4^4 = e^{-2iu}\frac{\sin(\xi-u)}{\sin(\xi+u)}.
\end{aligned}
\tag{22}
$$

- For the K-matrix $K^{(2)}(u)$, the 10 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}}{e^{2iu}-e^{\rho}}\left(1+e^{\rho}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^1 = -\frac{e^{-2i\eta\lambda_1+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_3^1 &= \frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_4^1 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2+\lambda_3)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^2 = 1,\\
K(u)_1^3 &= -\frac{e^{2i\eta(\lambda_1+\lambda_2)+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\ K(u)_2^3 = \frac{e^{2i\eta(\lambda_2)+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\\
K(u)_3^3 &= -\frac{1}{e^{2iu}-e^{\rho}}\left(e^{\rho}+\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_4^3 = \frac{e^{-2i\eta\lambda_3-i(u-\xi)+\rho}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\\
K(u)_4^4 &= e^{-2iu}\frac{\sin(\xi-u)}{\sin(\xi+u)}.
\end{aligned} \tag{23}
$$

- For the K-matrix $K^{(3)}(u)$, the 10 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}}{e^{2iu}-e^{\rho}}\left(1+e^{\rho}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^1 = -\frac{e^{-2i\eta\lambda_1+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_3^1 &= \frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\\
K(u)_4^1 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2+\lambda_3)+\rho}}{e^{2iu}-e^{\rho}}\left(1+e^{2iu}\frac{\sin(u-\xi)}{\sin(u+\xi)}\right),\ K(u)_2^2 = K(u)_3^3 = 1,\\
K(u)_1^4 &= \frac{e^{2i\eta(\lambda_1+\lambda_2+\lambda_3)+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\ K(u)_2^4 = -\frac{e^{2i\eta(\lambda_2+\lambda_3)+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\\
K(u)_3^4 &= \frac{e^{2i\eta\lambda_3+i(u+\xi)}\sin 2u}{(e^{2iu}-e^{\rho})\sin(u+\xi)},\ K(u)_4^4 = -\frac{1}{e^{2iu}-e^{\rho}}\left(e^{\rho}+\frac{\sin(u-\xi)}{\sin(u+\xi)}\right).
\end{aligned} \tag{24}
$$

- For the K-matrix $K^{(4)}(u)$, the 7 non-vanishing matrix elements $K(u)_j^k$ are given by:

$$
\begin{aligned}
K(u)_1^1 &= \frac{e^{2iu}-e^{4iu+\rho}}{e^{2iu}-e^{\rho}},\ K(u)_2^1 = \frac{e^{-2i\eta\lambda_1+\rho}\,(e^{4iu}-1)}{e^{2iu}-e^{\rho}},\\
K(u)_3^1 &= -\frac{e^{-2i\eta(\lambda_1+\lambda_2)+\rho}\,(e^{4iu}-1)}{e^{2iu}-e^{\rho}},\ K(u)_4^1 = \frac{e^{-2i\eta(\lambda_1+\lambda_2+\lambda_3)+\rho}\,(e^{4iu}-1)}{e^{2iu}-e^{\rho}},\\
K(u)_2^2 &= K(u)_3^3 = K(u)_4^4 = 1.
\end{aligned} \tag{25}
$$

In summary, we have presented the explicit matrix expressions of the non-diagonal K-matrices obtained in [4] for the trigonometric $A_{n-1}^{(1)}$ vertex model. From these results, it is easily seen that the K-matrices $K^{(l)}(u)\,(l=1,\ldots,n-1)$ depend on $n+1$ continuous free

parameters and have $3n-2$ non-vanishing matrix elements, and that the K-matrix $K^{(n)}(u)$ depends on n continuous free parameters and has $2n-1$ non-vanishing matrix elements.

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